Black hole state counting in loop quantum gravity

P. Mitra*
Saha Institute of Nuclear Physics,
1/AF Bidhannagar, Calcutta 700064

Abstract

Counting of microscopic states of black holes is discussed within the framework of loop quantum gravity. There are two different ways, one allowing for all spin states and the other involving only pure horizon states. The number of states with a definite value of the total spin is also found.

1 Introduction

The framework of quantum gravity known as loop quantum gravity has been able to yield a detailed counting of microscopic quantum states corresponding to a black hole. A start was made in [1] in the direction of quantizing a black hole characterized by an isolated horizon. The quantum states arise when the cross sections of the horizon are punctured by spin networks. The spin quantum numbers j, m, which characterize the punctures, then label the quantum states. The entropy is obtained by counting the possibilities of such labels that are consistent with a fixed area of the cross section [1].

A calculation of the entropy was carried out in [2] using a recursion relation technique. In [3], (see also [4] on this issue) a combinatorial method was introduced, which in addition to counting states also gives the dominant configuration of spins, namely the configuration yielding the maximum number of states. However, the two calculations give different results. The difference is due to the fact that while [2] takes into account only the spin projection (m) labels of the microstates, thus counting what may be called the pure *horizon* states, [3] and [4] take into account the spin j, which is relevant for the eigenvalues of the area operator, as well as the m-labels. There are two constraints to be satisfied. While one of them, the spin projection constraint, can be expressed solely in terms of the m-labels, the other constraint involving the area of the horizon, explicitly uses the j-labels.

*parthasarathi.mitra@saha.ac.in

2 Counting of states

We temporarily use units such that $4\pi\gamma\ell_P^2 = 1$, where γ is the so-called Barbero-Immirzi parameter involved in the quantization and ℓ_P the Planck length. Setting the classical area A of the horizon equal to the eigenvalue (for a specific spin configuration of punctures on the horizon) of the area operator we find

$$A = 2\sum_{j,m} s_{j,m}\sqrt{j(j+1)}, \tag{1}$$

where $s_{j,m}$ is the number of punctures carrying spin quantum numbers j, m. Such a spin configuration will be admissible if it obeys (1) together with the *spin projection constraint*

$$0 = \sum_{j,m} m s_{j,m} \ . \tag{2}$$

The total number of quantum states for these configurations is

$$d_{s_{j,m}} = \frac{(\sum_{j,m} s_{j,m})!}{\prod_{j,m} s_{j,m}!} \ . \tag{3}$$

To obtain the dominant permissible configuration that contributes the largest number of quantum states, we maximize $\ln d_{s_{j,m}}$ by varying $s_{j,m}$ subject to the constraints using Stirling's approximation:

$$\begin{aligned} \ln d_{s_{j,m}} &= (\sum_{j,m} s_{j,m}) \ln \sum_{j,m} s_{j,m} - \sum_{j,m}(s_{j,m}\ln s_{j,m}), \\ \delta \ln d_{s_{j,m}} &= (\sum_{j,m} \delta s_{j,m}) \ln \sum_{j,m} s_{j,m} - \sum_{j,m}(\delta s_{j,m}\ln s_{j,m}). \end{aligned} \tag{4}$$

The condition for the maximum can be expressed in terms of two Lagrange multipliers λ, α:

$$\ln s_{j,m} - \ln \sum_{j,m} s_{j,m} = -2\lambda\sqrt{j(j+1)} - \alpha m, \tag{5}$$

whence

$$\frac{s_{j,m}}{\sum s_{j,m}} = e^{-2\lambda\sqrt{j(j+1)}-\alpha m} \ . \tag{6}$$

Consistency requires that λ and α be related to each other by

$$\sum_j e^{-2\lambda\sqrt{j(j+1)}} \sum_m e^{-\alpha m} = 1. \tag{7}$$

In order that (6) satisfies the spin projection constraint, we need $\sum_m m e^{-\alpha m} = 0$ for each j, which essentially implies $\alpha = 0$. Therefore, the consistency condition becomes

$$\sum_{j,m} e^{-2\lambda\sqrt{j(j+1)}} = 1. \tag{8}$$

Numerical solution of this equation yields $\lambda = 0.861$. Note that each $s_{j,m}$ is proportional to the area A because of the area constraint. Further, in general,

$$\ln d_{s_{j,m}} = \lambda A + \alpha \sum_{j,m} s_{j,m} m, \tag{9}$$

in which the last term vanishes in the present situation because of the spin projection condition, but will appear later.

The total number of quantum states for all permissible configurations is clearly $d = \sum_{s_{j,m}} d_{s_{j,m}}$. To estimate d we expand $\ln d$ around the dominant configuration (6), which we shall denote by $\bar{s}_{j,m}$. Thus

$$\ln d = \ln d_{\bar{s}_{j,m}} - \frac{1}{2} \sum \delta s_{j,m} K_{j,m;j'm'} \delta s_{j'm'} + o(\delta s_{j,m}^2) \tag{10}$$

where $\delta s_{j,m} = s_{j,m} - \bar{s}_{j,m}$ and K is the symmetric matrix

$$K_{j,m;j'm'} = \delta_{jj'} \delta_{mm'} / \bar{s}_{j,m} - 1 / \sum_{k,l} \bar{s}_{k,l}. \tag{11}$$

The sum over each $\delta s_{j,m}$ can be approximated by a Gaussian integral. The eigenvalues of K are proportional to $1/A$, so each integration produces a factor $\sqrt{A}$. The number of these factors is two less than the number of $s_{j,m}$ because of the two constraints on the $\delta s_{j,m}$. On the other hand, we see from (3) that the combinatorial number contains one $\sqrt{A}$ for each $s_{j,m}$ in the denominator and one more in the numerator because

$$n! \approx \sqrt{2\pi n} (\frac{n}{e})^n. \tag{12}$$

In all, one factor of $\sqrt{A}$ survives in the denominator, so that

$$d = \frac{\text{constant}}{\sqrt{A}}\, e^{\lambda A}\ , \tag{13}$$

leading to the formula [3]

$$S = \lambda \frac{A}{4\pi\gamma\ell_P^2} - \frac{1}{2} \ln \frac{A}{4\pi\gamma\ell_P^2} \tag{14}$$

for entropy. The origin of the $\sqrt{A}$ in d or $\frac{1}{2}\ln A$ in $\ln d$ can be easily traced in this approach: it is the condition $\sum m s_{j,m} = 0$.

3 Counting of horizon states

The above calculation assumed that j is a relevant quantum number. An alternative procedure [1, 2], is to count the states of the horizon Hilbert space

alone. Here, following [5], we consider the number s_m of punctures carrying spin projection m, ignoring what spins j they are associated with. Clearly,

$$s_m = \sum_j s_{j,m}, \quad j = |m|, |m|+1, |m|+2, ... \tag{15}$$

For the s_m configuration the number of states is $d_{s_m} = (\sum_m s_m)!/\prod_m s_m!$ and the total number of states is obtained by summing over all configurations. As in the earlier case, the sum can be approximated by maximizing $\ln d_{s_m}$ subject to the two conditions. The constrained extremization conditions for variation of $s_{j,m}$ are

$$-\big[\ln \frac{s_m}{\sum_m s_m} + 2\lambda\sqrt{j(j+1)} + \alpha m\big] = 0. \tag{16}$$

All these equations cannot hold for arbitrary j even for a fixed m, because inconsistencies will arise for nonzero λ. In fact, for any fixed m the above equality can be valid for at most one j – say $j(m)$. For $j \neq j(m)$, the first derivative becomes nonzero. Such a situation can arise if and only if $\ln d_{s_m}$ is maximized at the boundary (in the space of all permissible configurations) for all $j \neq j(m)$ and at an interior point for $j = j(m)$. This means that for the dominant configuration, $s_{j,m} = 0$ for all $j \neq j(m)$: the corresponding first derivative is then only required to be zero or negative because in any variation $s_{j,m}$ can only increase from its zero value. Thus, $s_m = s_{j(m),m}$ for the dominant configuration and further, for $\lambda > 0$, $j(m) = j_{\min}(m)$, the minimum value for the m. For $m \neq 0$, we have $j_{\min}(m) = |m|$.

Then (16) gives

$$\frac{s_m}{\sum_m s_m} = e^{-2\lambda\sqrt{j_{\min}(m)(j_{\min}(m)+1)} - \alpha m} . \tag{17}$$

As before, $\alpha = 0$ because of the spin projection constraint.

The configuration (17) implies that the entropy is given by (14) in terms of λ, which is now determined by the altered consistency relation

$$1 = \sum_j 2e^{-2\lambda\sqrt{j(j+1)}}. \tag{18}$$

Note that for λ zero or negative, such relations would be impossible to satisfy, hence no such solutions exist.

This equation for λ aagrees with that of [2].

4 Vanishing total spin projection?

We have imposed the condition of vanishing spin projection in the above calculation. It is interesting to fix the total spin projection to some value and see how the number of states changes with this quantity. Thus we set

$$\sum_{j,m} m s_{j,m} = p. \tag{19}$$

The main difference with earlier equations will arise from the fact that α will no longer vanish. Let us introduce

$$F(\lambda, \alpha) \equiv \sum_{j,m} e^{-2\lambda\sqrt{j(j+1)}-\alpha m}. \tag{20}$$

Then we have the conditions

$$\begin{aligned} F(\lambda, \alpha) &= 1, \\ \frac{p}{A} &= \frac{\frac{\partial F}{\partial \alpha}}{\frac{\partial F}{\partial \lambda}}. \end{aligned} \tag{21}$$

These two equations determine λ, α in terms of $\frac{p}{A}$. On the basis of what we already know, we can write the general equation

$$\ln d = \lambda A + \alpha p - \frac{1}{2} \ln A. \tag{22}$$

Now

$$\lambda A + \alpha p = A(\lambda + \alpha \frac{\frac{\partial F}{\partial \alpha}}{\frac{\partial F}{\partial \lambda}}) = A(\lambda(\alpha) - \alpha \frac{d\lambda}{d\alpha}), \tag{23}$$

where $\lambda(\alpha)$ is understood to be the solution of $F = 1$. If $\frac{p}{A}$ is small, α can be taken to be small, and by Taylor expansion of $\lambda(\alpha)$ about $\alpha = 0$, we find

$$\lambda A + \alpha p \approx A(\lambda(0) - \frac{\alpha^2}{2} \frac{d^2\lambda}{d\alpha^2}|_{\alpha=0}). \tag{24}$$

Note that

$$\frac{d^2\lambda}{d\alpha^2}|_{\alpha=0} = -\frac{\frac{\partial^2 F}{\partial \alpha^2}}{\frac{\partial F}{\partial \lambda}}|_{\alpha=0} = \frac{\sum_{j,m} m^2 e^{-2\lambda\sqrt{j(j+1)}}}{2\sum_{j,m} \sqrt{j(j+1)} e^{-2\lambda\sqrt{j(j+1)}}} = k, \tag{25}$$

say, which is positive. Again, by expanding in α for small α, we find

$$\frac{p}{A} = \alpha \frac{\frac{\partial^2 F}{\partial \alpha^2}}{\frac{\partial F}{\partial \lambda}}|_{\alpha=0} = -\alpha k. \tag{26}$$

Hence,

$$\lambda A + \alpha p = A\lambda(0) - \frac{p^2}{2kA}, \tag{27}$$

and

$$\ln d = \lambda(0) A - \frac{p^2}{2kA} - \frac{1}{2} \ln A. \tag{28}$$

Note that $\lambda(0)$ here is the same as the λ of the earlier situation where $\alpha = 0$.

The number of states for a definite value of p is thus (cf. [2])

$$d(p) \sim \frac{\exp(\lambda(0)A - \frac{p^2}{2kA})}{\sqrt{A}} \approx \frac{\exp(\lambda(0)A)(1 - \frac{p^2}{2kA})}{\sqrt{A}}. \tag{29}$$

The number of states for total spin J and the same spin projection is then

$$d(J) - d(J+1) \sim (2J+1)\frac{\exp(\lambda(0)A)}{2kA\sqrt{A}}. \tag{30}$$

The number of states for (small) total spin J is

$$N(J) \sim (2J+1)^2 \frac{\exp(\lambda(0)A)}{2kA\sqrt{A}}. \tag{31}$$

In particular, for $J = 0$, this becomes

$$N(0) \sim \frac{\exp(\lambda(0)A)}{2kA^{3/2}}. \tag{32}$$

These may be compared with the results found in [6]. The reason for the disagreement between the coefficient of the log correction in the entropy found there and that in [7, 2, 3, 4] is seen to be that the *total spin* is involved in the former, while in the latter, following the loop quantum gravity literature, only the spin projection constraint is imposed.

Post script

However, it has now been suggested [8] that in an alternative quantization, at least for large area, the total spin J has to vanish.

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